General Change-of-Variables

<u>Thm:</u> Suppose g is a transformation whose Jacobian determinant is nonzero and that g transforms the region S in the uv plane onto the region R in the xy plane. Suppose that f is a continuous function on R. Then

$$\iint_R f(x,y)\,dxdy = \iint_S f(g(u,v))|\det J_g(u,v)|\,dudv$$

Notation reminder: The **Jacobian** $J_g(u,v)$ of the transformation $(x,y) = g(u,v)$ is the matrix

$$J_g(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and so the absolute value of its determinant is

$$|\det J_g(u,v)| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

The expression $|\det J_g(u,v)|$ should be treated as a function of the variables u, v.

<u>Ex 1:</u> Consider the transformation $x = u\cos v$ and $y = u\sin v$. Compute the Jacobian of this transformation and use it to express $\iint_R f(x,y)\,dxdy$ as an integral with respect to $dudv$.

Solution. We first compute the Jacobian of the transformation $g(u,v) = (x,y)$:

$$J_g(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix}$$

The absolute determinant of the transformation is

$$|\det J_g(u,v)| = |u\cos^2 v + u\sin^2 v| = |u| = u$$

since $u \geq 0$. Note that this agrees with the formula from the Polar Coordinates transformation. ■

In probability, we are often interested in the Jacobian of the inverse transformation $J_{g^{-1}}(x,y)$, which is

$$J_{g^{-1}}(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

and which has absolute determinant

$$|\det J_{g^{-1}}(x,y)| = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|$$

which is viewed as a function of (x,y).

But it turns out that the absolute determinant of $J_{g^{-1}}$ and of J_g are closely related:

$$|\det J_{g^{-1}}(x,y)| = \frac{1}{|\det J_g(g^{-1}(x,y))|}$$

Transformations of Random Variables

Suppose U, V are random variables with joint PDF $f_{U,V}(u,v)$ and support S. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible transformation, and define random variables X and Y by $(X,Y) = g(U,V)$. Then the joint PDF of X, Y is

$$f_{X,Y}(x,y) = f_{U,V}(g^{-1}(x,y))\frac{1}{|\det J_g(g^{-1}(x,y))|} = f_{U,V}(g^{-1}(x,y))|\det J_{g^{-1}}(x,y)|$$

with support $g(S)$.

Proof. Suppose $(X,Y) = g(U,V)$. Let $A \subset S$ and defined $B = g(A)$. We are interested in calculating $P((X,Y) \in V)$:

$$P((X,Y) \in B) = P(g(U,V) \in g(A)) = P((U,V) \in A) = \iint_A f_{U,V}(u,v)\, dudv = \iint_B f_{U,V}(g^{-1}(x,y))|\det J_{g^{-1}}(x,y)|\, dxdy$$

Therefore, the density function $f_{X,Y}(x,y)$ is $f_{U,V}(g^{-1}(x,y))|\det J_{g^{-1}}(x,y)|$. $\square$

Ex 2: Let U, V be iid $N(0,1)$ and define $X = U + V$ and $Y = U - V$. Find a formula for the joint PDF of X and Y.

Solution. Let $(x,y) = g(u,v) = (u+v, u-v)$. The inverse for this transformation is $g^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$. The Jacobian of the inverse is

$$J_{g^{-1}}(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and so the absolute Jacobian determinant of the inverse is

$$|\det J_{g^{-1}}(x,y)| = \left| -\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{2} \right| = \frac{1}{2}$$

For extra practice, we can also compute the Jacobian of g:

$$J_g(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which has absolute determinant

$$|\det J_g(u,v)| = |-1-1| = 2$$

Next, we need to find the joint PDF of U, V. Since both U and V are iid $N(0,1)$, the joint PDF is the product of their marginal PDFs:

$$f_{U,V}(u,v) = f_U(u)f_V(v) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2} = \frac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)}$$

Substituting $(u,v) = g^{-1}(x,y)$, we obtain

$$f_{U,V}(g^{-1}(x,y)) = \frac{1}{2\pi}e^{-\frac{1}{2}\left(\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2\right)}$$

which after a little algebra simplifies to

$$f_{U,V}(g^{-1}(x,y)) = \frac{1}{2\pi}e^{-\frac{1}{4}(x^2+y^2)}$$

Using the change-of-variables formula, the joint PDF of X, Y is

$$f_{X,Y}(x,y) = f_{U,V}(g^{-1}(x,y))|\det J_{g^{-1}}(x,y)| = \frac{1}{2\pi}e^{-\frac{1}{4}(x^2+y^2)}\frac{1}{2}$$

Note that this joint density factors as

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi \cdot 2}}e^{-\frac{x^2}{4}}\frac{1}{\sqrt{2\pi \cdot 2}}e^{-\frac{y^2}{4}}$$

which shows that X and Y are independent, and moreover, that the marginal of distribution of each is $N(0,2)$. $\blacksquare$