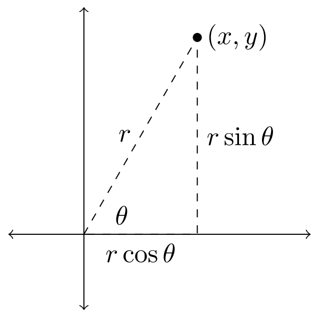
Polar Coordinates

Every point in the plane can be represented in two ways: in cartesian coordinate form as (x, y) or in polar coordinate form as (r, θ). To convert between forms, we use trigonometry:



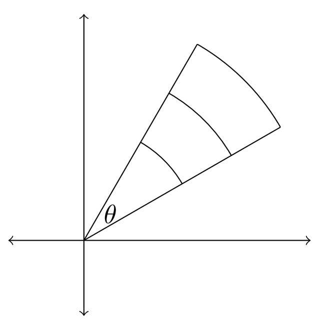
$$x = r\cos\theta \quad q = y\sin\theta$$

Or the reverse:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(x/y) \quad \text{if } x, y > 0$$

We can use this fact to our advantage when calculating double integrals that involve either circular domains, or rotationally symmetric functions. However, we must be careful when converting from cartesian to rectangular coordinates that we preserve areas in the plane:

If $dxdy$ denotes the area of a very small rectangle in cartesian coordinates, the area of the same rectangle in polar coordiantes is $r\,drd\theta$.



Suppose $\mathcal{R}$ is a region in the plane specified by a pair of intervals in Polar coordinates: $a \le r \le b$ and $\alpha \le \theta \le \beta$. To calculate the volume under the function $f(x, y)$ over $\mathcal{R}$:

$$\iint_{\mathcal{R}} f(x, y)\,dxdy = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\,r\,drd\theta$$

Ex 1: Suppose X, Y are iid $N(0, 1)$. Find the probability that $X^2 + Y^2 \le 1$.

General Change-of-Variables

Consider $\int_0^1 x^2\sqrt{1-x^2}\,dx$. Using the substitution $x = \sin u$, we obtain

$$\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sin^2\theta\sqrt{1-\sin^2 u}\cos u\,du$$

When performing the substitution, we made three separate conversions: the integrand, the bounds of integration, and the differential.

We now consider general transformations in two variables.

Consider an integral of the form $\iint_D f(x,y)\,dxdy$, and suppose we wish to make a substitution by the transformation

$$(x,y) = g(u,v) \qquad \text{where } x = x(u,v) \quad y = y(u,v)$$

As before, we need to make three separate conversions: the integrand, the bounds of integration, and the differential.

In general, when making the substitution $(x,y) = g(u,v)$, the region D will be the image of the region S under the transformation g. If g has an inverse g^{-1}, then S will be the image of S under the transformation g^{-1}. In practice, this amounts to expressing the boundaries of the region of interest in the new coordinate system.

Ex 2: Suppose R is the parallelogram $(0,0),(5,0),(5/2,5/2),(5/2,-5/2)$. And let g be the transformation $x = 2u+3v$ and $y = 2u-3v$. Find the image of R under g^{-1}.

From Calculus II, it turns out that if $dxdy$ is the area of a small rectangular region in Cartesian Coordinates, then the area of the same region in (u,v) coordinates is

$$dxdy = |\det J_g(u,v)|\,dudv$$

where $J_g(u,v)$ is the **Jacobian matrix** of the transformation g given by

$$J_g(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Thm: Suppose g is a transformation whose Jacobian determinant is nonzero and that g transforms the region S in the uv plane onto the region R in teh xy plane. Suppose that f is a continuous function on R. Then

$$\iint_R f(x,y)\,dxdy = \iint_S f(g(u,v))|\det J_g(u,v)|\,dudv$$

Ex 3: Show that the polar coordinates formula is a consequence of this theorem.

Ex 4: Suppose R is the parrellelogram in xy plane with vertices $(0,0),(5,0),(5/2,5/2),(5/2,-5/2)$. Evaluate $\iint_R x+y\,dxdy$ using the transformation $x = 2u+3v$ and $y = 2u-3v$.