Taylor Series

Motiviation

Recall the formula for the geometric series

$$\sum_{k=0}^{\infty} ax^k$$

which converges to $\frac{a}{1-x}$ if $|x| < 1$ and diverges if $|x| \geq 1$.

Here, we are treating the ratio x as a fixed but unknown constant. But instead of thinking of x as a constant, we could think of it a variable, and say that the geometric series defines a function

$$f(x) = \sum_{k=0}^{\infty} ax^k = \frac{a}{1-x}$$

with domain $(-1, 1)$.

This allows us to analyze the function f from two distinct perspectives: 1) using a compact formula like $\frac{a}{1-x}$ and (2) using an infinite sum of polynomials.

Power Series

Def: A **power series** has the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

where a_k is a coefficient that may depend on k, but does not depend on x.

For each value of x, we obtain a different infinite series of terms, which may either converge or diverge. When we treat the power series as a function of x, the domain of the function is all values of x where the power series converges.

It turns out that the domain of convergence of a power series is always a symmetric interval centered at 0 (for example, the interval $(-1, 1)$). The distance from the endpoint of the interval to 0 is called **the radius of convergence**.

In this class, we won't worry too much about finding the radius of convergence of specific powere series. We just need to be aware that not every power series converges for all real numbers.

Representing Functions with Power Series

Recall from calculus that for $n \geq 1$,

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \int x^n \, dx = \frac{1}{n+1} x^n$$

which allow us to easily take derivatives and integrals of polynomials by differentiating / integrating term-by-term. Since power series are really just 'infinite' polynomials, it turns out that same rule is true for power series as well.

Thm: If the power series $f(x) = \sum a_k x^k$ has radius of convegence $R > 0$, then f is differentiable and integrable on the interval $(-R, R)$ and

1. $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$
2. $\int f(x) \, dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$

Ex 1: Represent the function $g(x) = \frac{1}{(1-x)^2}$ as a power series.

Here is a related problem. Which functions can be represented by a power series? And how do we determine the coefficients of that series?

Suppose f is a function represented by a power series with

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \qquad |x| < R$$

Note that when $x = 0$,

$$f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \dots = a_0$$

And now, since we can differentiate power series,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

so

$$f'(0) = a_1 + a_2 \cdot 0 + 3a_3 \cdot 0 + \dots = a_1$$

and

$$f''(0) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \dots$$

so that

$$f''(0) = 2a_2 + 3 \cdot 2a_3 \cdot 0 + 4 \cdot 3 \cdot a_4 \cdot 0 + \dots$$

In general,

$$f^{(n)} = n! a_n + (n+1)(n)(n-1)\dots 2a_{n+1} x + \dots$$

so that

$$f^{(k)}(0) = k! a_k$$

Solving for a_k gives

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Thm: If f has a power series representation at a, that is

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad |x-a| < R$$

then the coefficients are given by

$$a_n = \frac{f^{(n)}(a)}{n!}$$

This series is called the **Taylor series** for f centered at a.

In the special case when $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and is called the Maclaurin series for f.

Note: we have shown that if f is represented by a power series, then its power series must take a particular form. But it is not always the case that a function is equal to its Taylor series.

Ex 2: We know that the function $f(x) = \frac{1}{1-x}$ is represented by the power series $1 + x + x^2 + \dots$. Verify that the coefficients of this power series are the same as for the Macluarin series for f.

Ex 3: Find the Macluarin series for $f(x) = e^x$.

Ex 4: Use the Taylor Series representation for $f(x) = \ln(1-x)$ to find the values of $f^{(k)}(0)$ for all values of k.