For extra practice, several additional review problems are printed below. Solutions to these problems can be found on the exams page of the course website. I've tried to sort the problems by whether I think they are more appropriate as an in-class question or as a take-home question. While these questions are representative of the typical scope and difficulty of individual exam questions, this review is not comprehensive, nor does it necessarily represent the total amount of time available for the exam.

In-class Exam.

(1) Determine whether the following statement is always true or sometimes false (If true, explain why. If false, give a counter-example):

If A, B, C are events so that A and B are conditionally independent given C, then A and B are independent.

(2) Suppose $X \sim \text{Bin}(8, 0.25)$ and let $Y = 2X + 1$. Find and simplify a formula for the PMF of Y. Then compute the expected value of Y.

(3) A box has three coins in it. One has heads on both sides, one has tails on both sides, and one is a fair coin. A coin is selected uniformly at random and flipped twice.

(a) If the result of the first flip is heads, what is the probability that the coin is two-headed?
(b) If the result of the first flip is heads, what is the probability that the second flip is also heads?
(c) If the result of both flips are heads, what is the probability that the coin is two-headed?

(4) Suppose A and B are events. Use axioms of probability to show that if $A \subseteq B$, then $P(A) \leq P(B)$.

(5) Two teams, the Alligators and the Bears, play a series of games, where the Alligators have probability p of winning each game (independently). The winner of the series is the first team to win 2 more games than their opponent. Compute the expected number of games in the series.

(6) Suppose A and B are events with $0 < P(B) < 1$. Show that if $P(A|B) > P(A|B^c)$, then

$$P(A|B) > P(A) > P(A|B^c).$$

(7) Define a function p on the set $D = \{1, 2, \ldots, 9\}$ by

$$p(k) = \log_{10}(k + 1) - \log_{10}(k)$$

(a) Verify that p is a valid PMF function.
(b) Suppose X is a random variable with PMF p. Find a formula for the CDF F of X.
(c) Let $Y \sim \text{DUnif}(D)$, independent of X. What is the probability that $Y = X$?

(8) On a certain statistics test (not necessarily this one!), 10 out of 100 key terms will be randomly selected to appear on an exam. A student then must choose 7 of these 10 to define. Since the student knows the format of the exam in advance, the student is trying to decide how many of these terms to memorize.

(a) Suppose the student studies s key terms, where s is an integer between 0 and 100. What is the distribution of X? Give the name and parameters of the distribution, in terms of s.
(b) Write an expression involving the explicit formula for the PMF or CDF of the named distribution for the probability that the student knows at least 7 of the 10 key terms, assuming the student studies $s = 75$ key terms.

Take-home Exam.

(1) A box contains a total of N marbles, colored blue, white and orange. Let b, w, o denote the proportions of blue, white and orange marbles, respectively.
   (a) Marbles are drawn 1-by-1 randomly with replacement. Find the probability that the first time a blue marble is drawn is before the first time a white marble is drawn.
   (b) Suppose instead that marbles are drawn 1-by-1 randomly without replacement. Find the probability that the first time a blue marble is drawn is before the first time a white marble is drawn.

(2) Suppose you are answering a multiple-choice problem on an exam, and have to choose one of n options (exactly one of which is correct). Let K be the event that you know the answer and let R be the event that you get the problem right (either through knowledge or through lucky guessing). Suppose that if you know the right answer, you will definitely get the problem right, but if you do not know the answer, you will guess completely randomly. Let $P(K) = p$.
   (a) Find $P(K|R)$ in terms of p and n.
   (b) Show that $P(K|R) \geq p$ and explain why this makes sense intuitively. When (if ever) does $P(K|R) = p$?

(3) Suppose n individuals in a large population have a rare disease. Two medical tests are available to detect the disease. Suppose the first test detects the disease with probability p_1, while the second test detects it with probability p_2. Assume that detection is independent between individuals, as well as between tests. Let X_1 be the number of individuals the first test detects, let X_2 be the number the second test detects, and let X be the number detected by at least one of the tests.
   (a) Find the distribution of X.
   (b) Assume for this part that $p_1 = p_2$. Find the conditional distribution of X_1 given that $X_1 + X_2 = t$.

(4) Consider a daily lottery in which 5 of the integers from 1 to 50 are chosen without replacement.
   (a) Find the probability that you guess exactly 3 numbers correctly, given that you guess at least 1 of the numbers correctly.
   (b) Find an exact expression for the expected number of lotteries needed so that all $\binom{50}{5}$ possible lottery outcomes have occurred.

(5) A fair 6-sided die is rolled 6 times. What is the probability that at least 1 of the 6 values never appears? First find an exact formula for the solution, and then verify by writing an R function and performing a simulation to estimate the probability.