Axioms of Probability

1. Consider the sample space $S = \{1, 2, \ldots, \}$ consisting of the positive integers.
   (a) Does there exist a probability function P so that $P(\{n\}) = P(\{m\})$ for all positive integers n, m? Why or why not?
   (b) Give an example of one probability function P that **is** defined on S.
   (c) Does there exist a probability function P and an event $A \subset S$ with infinitely many elements so that $P(\{n\}) = P(\{m\})$ for all $n, m \in A$? If so, what has to be true about $P(A)$?

2. The **Bonferroni Inequality** gives a lower bound on the probability that multiple events simultaneously occur, and is an important result when performing statistical tests involving multiple comparisons. The identity states: If $A_1, \ldots, A_n$ are a finite list of events, then

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

   (a) Show that the Bonferroni Inequality holds when $n = 2$ in two ways: first with with a picture, and second with a rigorous mathematical proof.
   (b) Nate is interested in estimating the average height and weight of a population of fox squirrels and plans to do so by collecting a random sample of 10 squirrels. Suppose he has a procedure that for 95% of all samples will produce an estimated height that is within 5cm of the true average height of the population, and that for 95% of all samples will produce an estimated weight that is within 20g of the true average weight of the population.

   Based on the Bonferroni inequality, what is the smallest the probability can be that a single random sample produces an estimated height within 5cm of the true average height **and** an estimated weight within 20g of the true average weight?

3. (*) Suppose Oliver has a belief system assigning a number $P_o(A)$ between 0 and 1 to every event $A \subset S$ for some sample space S. This represents Oliver's degree of belief about how likely A is to occur. For every event A, Oliver is willing to pay $P_o(A)$ dollars to buy from you a certificate that says

   > "The owner of this certificate can redeem it from the seller for \$1 if A occurs, and for \$0 if A does not occur."

   Likewise, Oliver is willing to sell such a certificate at the same price. In fact, he will buy/sell any number of certificates for this price.

   However, as a four year old, Oliver stubbornly refused to accept the axioms of probability. In particular, there are a particular pair of disjoint events A and B with

$$P_o(A \cup B) \neq P_o(A) + P_o(B).$$

   Suppose Oliver was gifted \$10 on his birthday. Show that there is a sequence of transactions that Oliver is willing to make that will **guarantee** he will lose all \$10. (You can assume that after all certificates have been bought and sold, we know for certain whether event A and whether event B occurred).

(*) Indicates problems that will also be collected for homework.