A process of arrivals in continuous time is called a *Poisson process* with rate λ if the following two conditions hold:

- The number of arrivals that occur in an interval of length t is a $\text{Pois}(\lambda t)$ random variable
- The number of arrivals that occur in disjoint intervals are independent of each other. For example, the number of arrivals in the interval $(0, 10]$, $(10, 12]$ and $(15, 100]$ are independent.

We'll now define a few variables related to this process:

- For each real number $t \geq 0$, let N_t be the number of arrivals in the interval $(0, t]$; then $N_t \sim \text{Pois}(\lambda t)$.
- For $s < t$, the variable $N_t - N_s$ counts the number of arrivals in the interval $(s, t]$; then $N_t - N_s \sim \text{Pois}(\lambda(t - s))$.
- For each integer $k \geq 1$, let T_j the time of the jth arrival; we will show $T_j \sim \text{Gamma}(j, \lambda)$.
- For each integer $k \geq 1$, let R_j be the time between the kth and $(k-1)$th arrival; we will show $R_j \sim \text{Expo}(\lambda)$.

1. The following exercises outline a proof that $T_j \sim \text{Gamma}(j, \lambda)$
   (a) Find the CDF for T_1 using the fact that $P(T_1 \leq t) = P(N_t \geq 1)$. Then calculate the PDF of T_1.
   (b) Express the probability $P(T_j > t)$ as a finite sum, using the fact that $P(T_j > t) = P(N_t < j)$.
   (c) Use your previous answer to write the CDF of T_j as a **finite** sum.
   (d) Differentiate and use the product rule to write the PDF of T_j as a finite sum. Be careful about the derivative of the constant term.
   (e) Explicitly write out the terms of the sum for the PDF of T_2 and T_3. What pattern do you notice?
   (f) Make a conjecture for how to simplify the sum for the PDF of T_j.
   (g) Based on your formula in the previous part, what is the name of the distribution of T_j?
2. The following exercises outline a proof that $R_j \sim \text{Expo}(\lambda)$.
   (a) Let j be a positive integer. Express the event $R_j > t$ as an event involving T_j and T_{j-1}.
   (b) Let $f_{j-1}(s)$ be the marginal PDF of T_{j-1}. Express $P(R_j > t)$ as an integral, by conditioning on the event $T_{j-1} = s$ and using continuous LotP.
   (c) Rewrite the conditional probability inside your integral as a conditional probability involving N_{t+s} and N_s.
   (d) Use the fact that for a Poisson process, the number of arrivals in disjoint intervals are independent to simplify your integral.
   (e) Use the fact that f_{j-1} is a valid PDF and therefore, must integrate to 1, in order to calculate the value of the integral.