Change-of-Variables

1. (*) Suppose U and V are iid Expo(λ). In this problem, we will find the joint distribution of $U+V$ and $\frac{U}{U+V}$, as well as the marginal distribution of $\frac{U}{U+V}$.

   (a) Define a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) = g(u, v) = \left(u + v, \frac{u}{u+v}\right)$. Find a formula for the inverse transformation $(u, v) = g^{-1}(x, y)$.

   (b) Calculate the Jacobian $J_g(u, v)$ and make the substitution $(u, v) = g^{-1}(x, y)$.

   (c) Let $X = U + V$ and $Y = \frac{U}{U+V}$. Use the change-of-variables formula to express the joint PDF $f_{X,Y}$ of X, Y in terms of the joint PDF $f_{U,V}$ of U, V.

   (d) Based on your previous answer, are X and Y independent?

   (e) Find a formula for the marginal PDF of $Y = \frac{U}{U+V}$. What named distribution is this?