Universality of the Uniform

1. (*) The *Pareto distribution* with parameter $a > 0$ has PDF $f(x) = a/x^{a+1}$ for $x \geq 1$ (and 0 otherwise).

   (a) Find the CDF of a Pareto r.v with parameter a.

   (b) Pareto distributions are said to be *heavy-tailed*, which means they have relatively high probability of generating large values. For what values of a does a Pareto variable have a mean? A variance? Compute the mean and variance for those pareto variables where it makes sense to do so.

   (c) R does not have a formula for generating Pareto random variables (unlike `rbinom` for the binomial distribution). But R does have a function to generating Uniform random variable (`runif`). Explain how to use `runif` to generate 100 samples of a variable with the Pareto$-a$ distribution.

2. (a) Suppose we sample 99 times from a uniform distribution and let $u_1, u_2, \ldots, u_{99}$ be the values obtained from sampling. Sort these values in ascending order, and let b_1 be the smallest, b_2 be the second smallest, and so on. It turns out that the expected value of b_i is $\frac{i}{100}$. Without doing any rigorous calculation, give an intuitive explanation for why this makes sense.

   (b) Now, consider a continuous random variable X with CDF F and quantile function F^{-1}. Suppose we sample from the distribution of X a total of 99 times, and let $x_1, x_2, \ldots, x_{99}$ be the values obtained from sampling. Sort the x_i's in ascending order, and let a_1 be the smallest, a_2 be the second smallest, and so on. In your own words, explain what each of the following equations mean:

   $$a_2 = 3.2 \qquad x_2 = 13 \qquad F(10) = 0.62 \qquad F^{-1}(0.15) = 2$$

   (c) Suppose we evaluate the CDF F of X at each of the points $a_1, \ldots, a_{99}$. It turns out that the value of $F(a_i)$ Is approximately $\frac{i}{100}$. Give an intuitive explanation for why this makes sense, using Universality of the Uniform.

   (d) Similarly, suppose we evaluate the quantile function F^{-1} at each of the values $\frac{1}{100}, \frac{2}{100}, \ldots, \frac{99}{100}$. It turns out that $F^{-1}\left(\frac{i}{100}\right) \approx a_i$. Give an intuitive explanation for why this makes sense, using Universality of the Uniform.

   (e) Do you think it would still be true that $F(x_i) \approx \frac{i}{100}$? Or that $F^{-1}\left(\frac{i}{100}\right) \approx x_i$? Why?